Lecture-1

Biot-Savart Law: magnetic field of a current element

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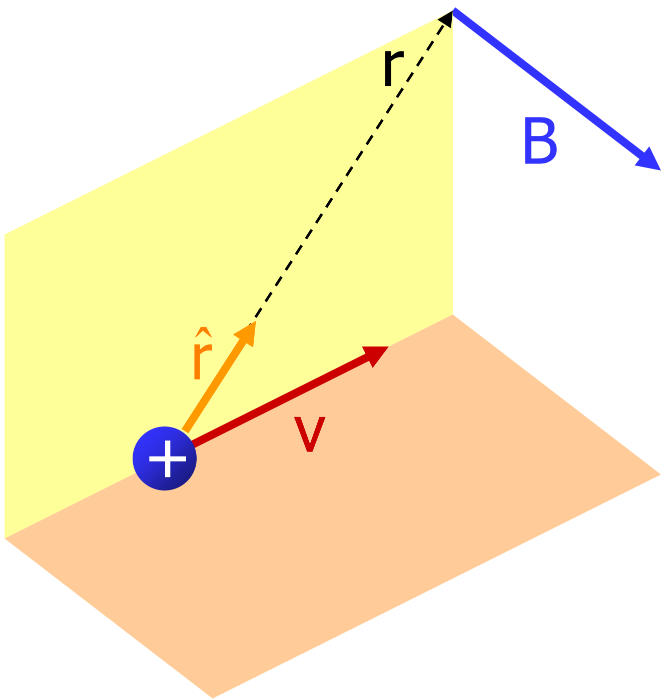
Biot-Savart Law: magnetic field of a current element

Let's start with the magnetic field of a moving charged particle.



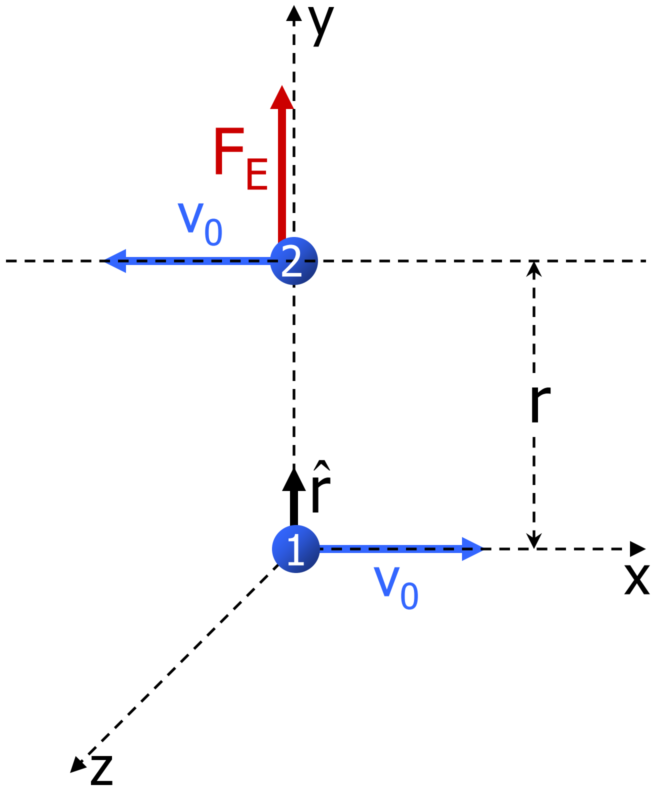
It is experimentally observed that a moving point charge q gives rise to a magnetic field

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}.$$

μ_0 is a constant, and its value is
$\mu_0 = 4\pi \times 10^{-7}$ T·m/A

Example: proton 1 has a speed v_0 ($v_0 << c$) and is moving along the x-axis in the +x direction. Proton 2 has the same speed and is moving parallel to the x-axis in the –x direction, at a distance r directly above the x-axis. Determine the electric and magnetic forces **on** proton 2 at the instant the protons pass closest to each other.

This is example 28.1 in your text.



The electric force is

$$\vec{F}_E = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

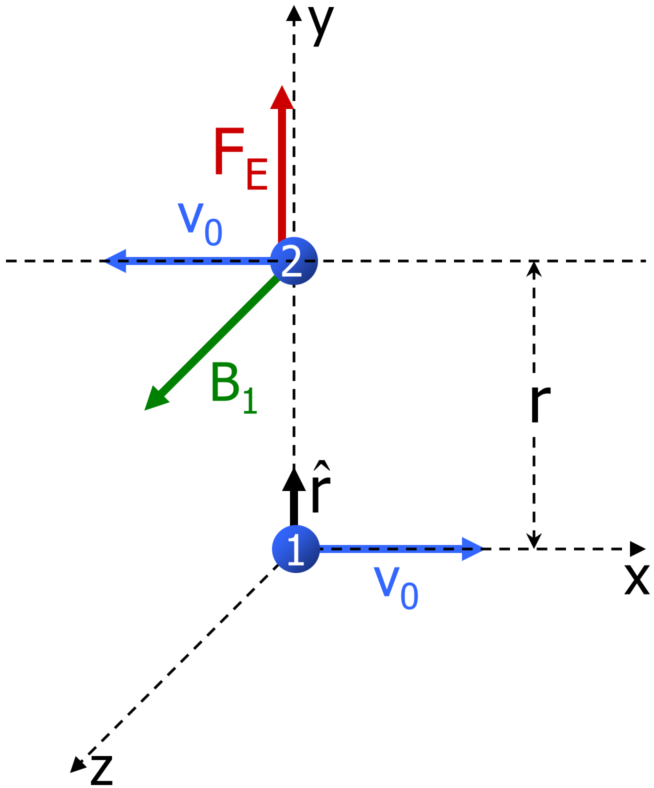
$$\vec{F}_E = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\hat{j}$$

At the position of proton 2 there is a magnetic field due to proton 1.

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{q_1\vec{v}_1 \times \hat{r}}{r^2}$$

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{ev_0\hat{i} \times \hat{j}}{r^2}$$

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{ev_0}{r^2}\hat{k}$$



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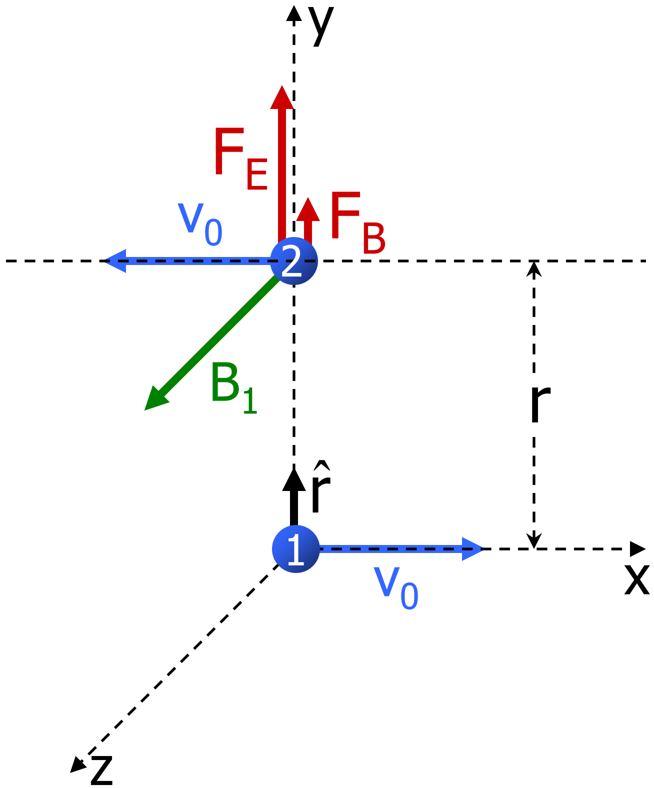
Proton 2 “feels” a magnetic force due to the magnetic field of proton 1.

$$\vec{F}_B = q_2 \vec{v}_2 \times \vec{B}_1$$

$$\vec{F}_B = e v_0 \left(-\hat{i}\right) \times \left(\frac{\mu_0}{4\pi} \frac{e v_0}{r^2} \hat{k} \right)$$

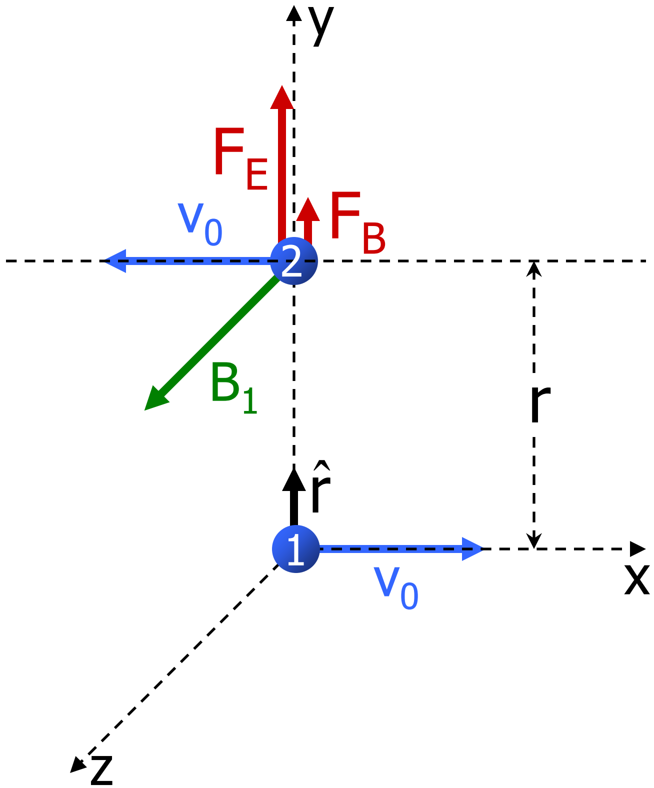
$$\vec{F}_B = \frac{\mu_0}{4\pi} \frac{e^2 v_0^2}{r^2} \hat{j}$$



Both forces are in the +y direction. The ratio of their magnitudes is

$$\frac{F_B}{F_E} = \frac{\left(\frac{\mu_0}{4\pi}\frac{e^2 v_0^2}{r^2}\right)}{\left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\right)}$$

$$\frac{F_B}{F_E} = \mu_0 \varepsilon_0 v_0^2$$



Later we will find that

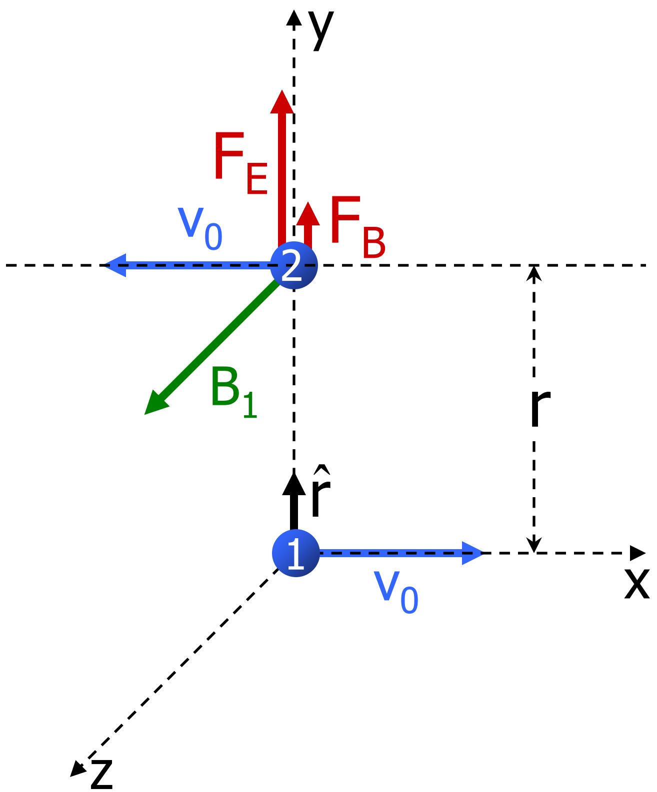
$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

Thus $\dfrac{F_B}{F_E} = \dfrac{v_0^2}{c^2}$

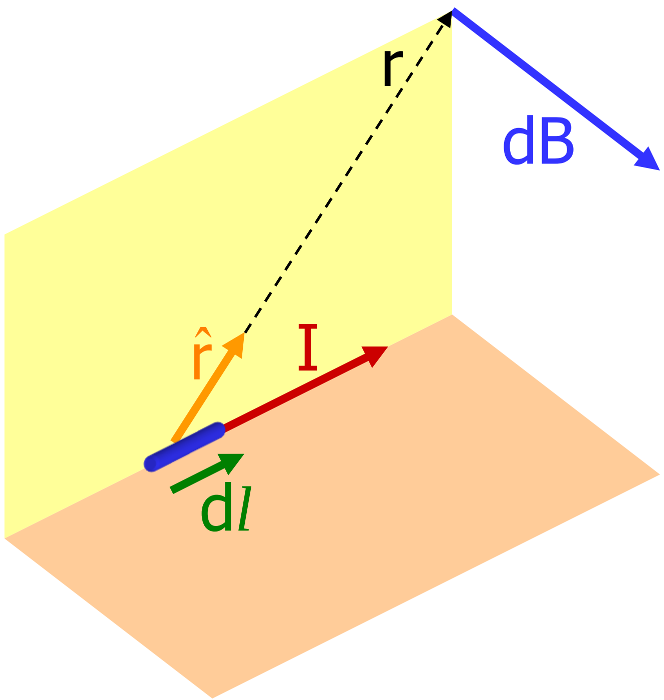
If $v_0 = 10^6$ m/s, then

$$\frac{F_B}{F_E} = \frac{\left(10^6\right)^2}{\left(3\times10^8\right)^2} = 1.11\times10^{-5}$$

Don’t you feel sorry for the poor, weak magnetic force?



From the equation for the magnetic field of a moving charged particle, it is "easy" to show that a current I in a little length d*l* of wire gives rise to a little bit of magnetic field.

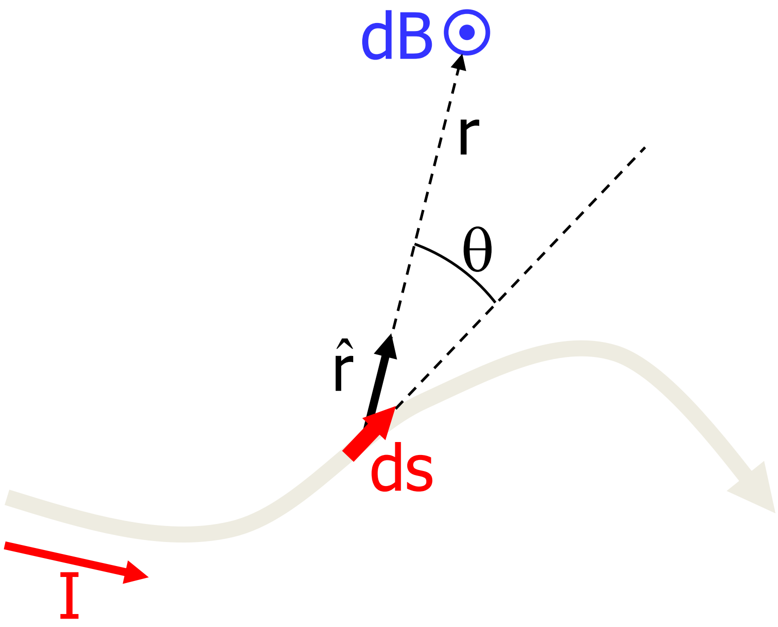


$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\ell} \times \hat{r}}{r^2}$$

The Biot-Savart Law

You may see the equation written using $\vec{r} = r\,\hat{r}$.

Applying the Biot-Savart Law

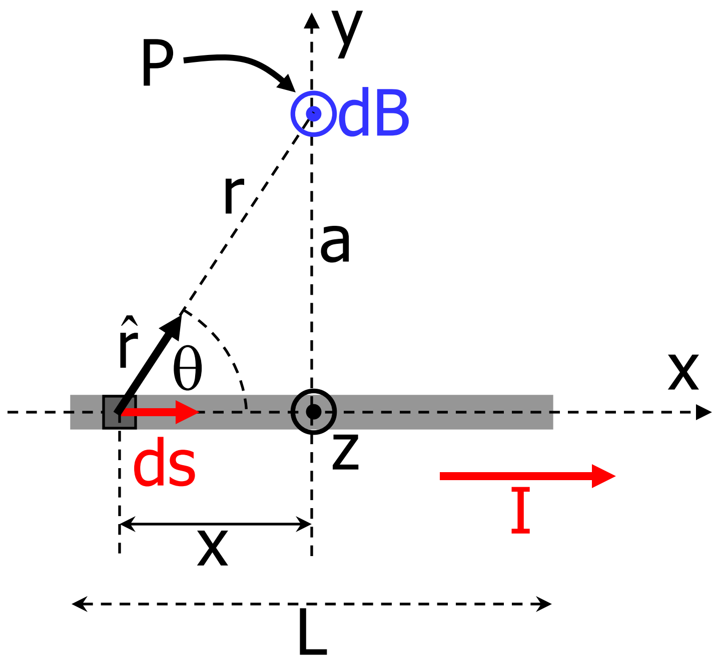


$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{s} \times \hat{r}}{r^2} \quad \text{where} \quad \hat{r} = \frac{\vec{r}}{r}$$

$$dB = \frac{\mu_0}{4\pi}\frac{I\, ds \sin\theta}{r^2}$$

$$\vec{B} = \int d\vec{B}$$

Example: calculate the magnetic field at point P due to a thin straight wire of length L carrying a current I. (P is on the perpendicular bisector of the wire at distance a.)



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{s} \times \hat{r}}{r^2}$$

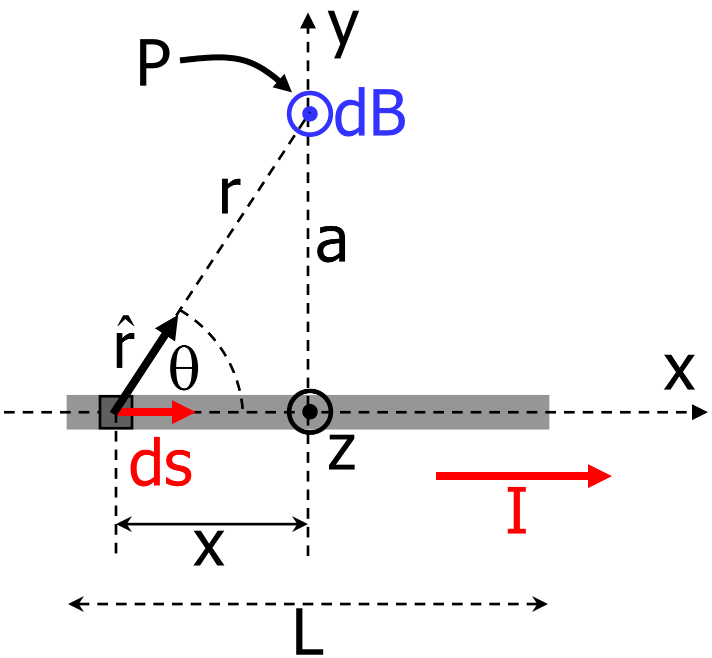
$$d\vec{s} \times \hat{r} = ds \sin\theta\, \hat{k}$$

$$dB = \frac{\mu_0}{4\pi} \frac{I\, ds \sin\theta}{r^2}$$

ds is an infinitesimal quantity in the direction of dx, so

$$dB = \frac{\mu_0}{4\pi} \frac{I\, dx \sin\theta}{r^2}$$

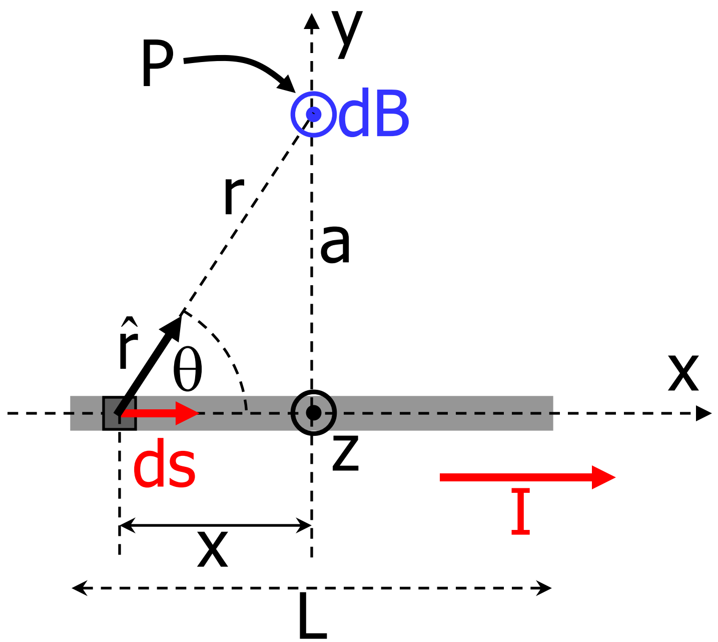
$$\sin\theta = \frac{a}{r} \qquad r = \sqrt{x^2 + a^2} \qquad dB = \frac{\mu_0}{4\pi}\frac{I\,dx\,\sin\theta}{r^2}$$



$$dB = \frac{\mu_0}{4\pi}\frac{I\,dx\,a}{r^3} = \frac{\mu_0}{4\pi}\frac{I\,dx\,a}{\left(x^2 + a^2\right)^{3/2}}$$

$$B = \int_{-L/2}^{L/2} \frac{\mu_0}{4\pi}\frac{I\,dx\,a}{\left(x^2 + a^2\right)^{3/2}}$$

$$B = \frac{\mu_0 I\,a}{4\pi}\int_{-L/2}^{L/2} \frac{dx}{\left(x^2 + a^2\right)^{3/2}}$$



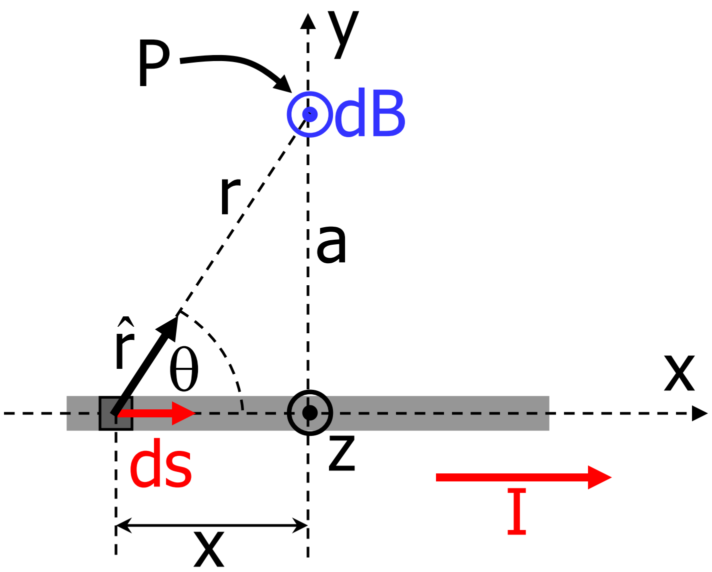
$$B = \frac{\mu_0 I\, a}{4\pi} \int_{-L/2}^{L/2} \frac{dx}{\left(x^2 + a^2\right)^{3/2}}$$

look integral up in tables, use the web,or use trig substitutions

$$\int \frac{dx}{\left(x^2 + a^2\right)^{3/2}} = \frac{x}{a^2\left(x^2 + a^2\right)^{1/2}}$$

$$B = \frac{\mu_0 I\, a}{4\pi} \left. \frac{x}{a^2\left(x^2 + a^2\right)^{1/2}} \right|_{-L/2}^{L/2}$$

$$= \frac{\mu_0 I\, a}{4\pi} \left[\frac{L/2}{a^2\left((L/2)^2 + a^2\right)^{1/2}} - \frac{-L/2}{a^2\left((-L/2)^2 + a^2\right)^{1/2}} \right]$$

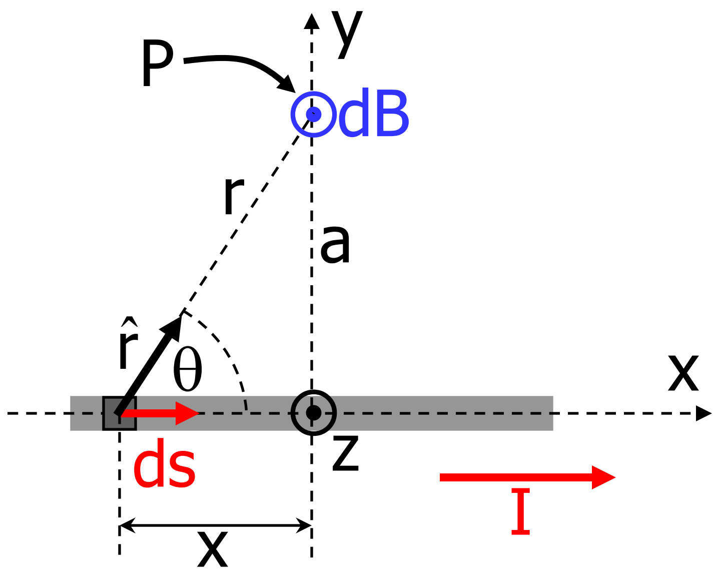


$$B = \frac{\mu_0 I\, a}{4\pi}\left[\frac{2L/2}{a^2\left(L^2/4 + a^2\right)^{1/2}}\right]$$

$$B = \frac{\mu_0 I\, L}{4\pi a}\frac{1}{\left(L^2/4 + a^2\right)^{1/2}}$$

$$B = \frac{\mu_0 I\, L}{2\pi a}\frac{1}{\sqrt{L^2 + 4a^2}}$$

$$B = \frac{\mu_0 I}{2\pi a}\frac{1}{\sqrt{1 + \dfrac{4a^2}{L^2}}}$$



$$B = \frac{\mu_0 I}{2\pi a} \frac{1}{\sqrt{1 + \frac{4a^2}{L^2}}}$$

When $L \to \infty$, $B = \dfrac{\mu_0 I}{2\pi a}$.

or $$B = \frac{\mu_0 I}{2\pi r}$$

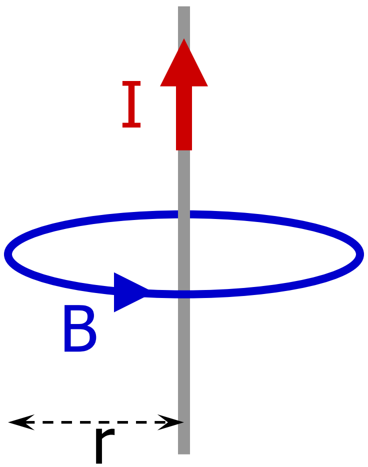
The r in this equation has a different meaning than the r in the diagram!

Magnetic Field of a Long Straight Wire

We've just derived the equation for the magnetic field around a long, straight wire*

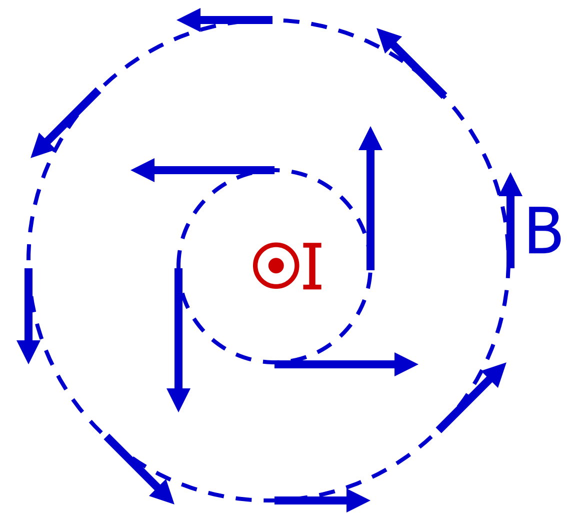
$$B = \frac{\mu_0 I}{2\pi r}$$

with a direction given by a "new" right-hand rule.



*Don't use this equation unless you have a long, straight wire!

Looking “down” along the wire:

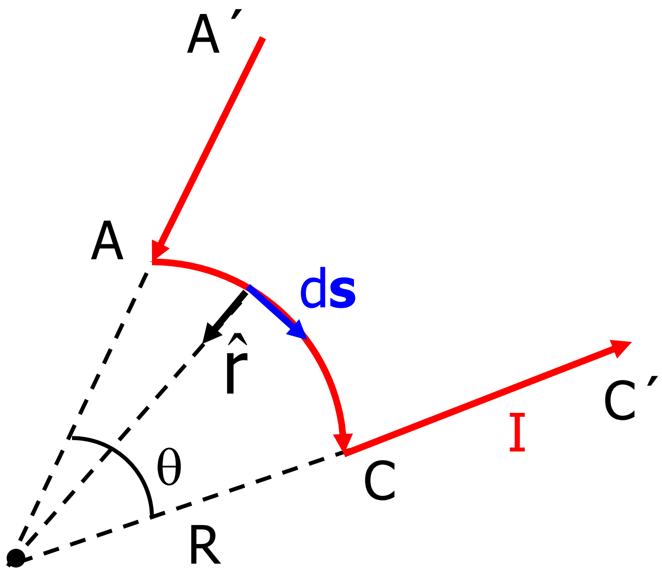


The magnetic field is not constant.

At a fixed distance r from the wire, the **magnitude** of the magnetic field is constant.

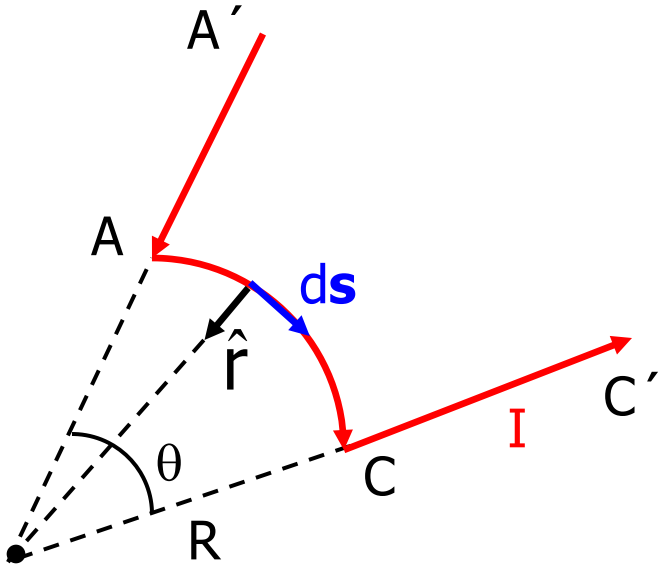
The magnetic field **direction** is always tangent to the imaginary circles drawn around the wire, and perpendicular to the radius “connecting” the wire and the point where the field is being calculated.

Example: calculate the magnetic field at point O due to the wire segment shown. The wire carries uniform current I, and consists of two straight segments and a circular arc of radius R that subtends angle θ.



Important technique, handy for exams:

The magnetic field due to wire segments A'A and CC' is zero because $d\vec{s}$ is parallel to $\hat{r}$ along these paths.



Important technique, handy for exams:

Along path AC, $\vec{ds}$ is perpendicular to $\hat{r}$.

$$d\vec{s} \times \hat{r} = -ds\,\hat{k}$$

If we use the "usual" xyz axes.

$$\left| d\vec{s} \times \hat{r} \right| = ds$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{s} \times \hat{r}}{r^2}$$

$$dB = \frac{\mu_0 I}{4\pi} \frac{ds}{R^2}$$

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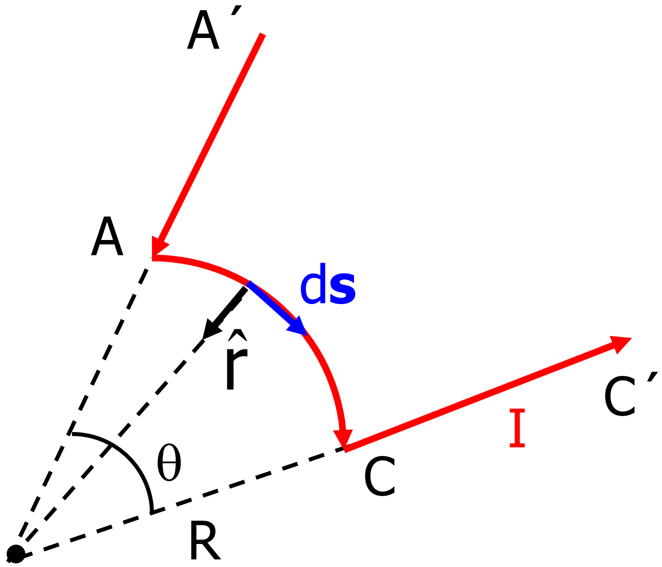


$$dB = \frac{\mu_0 I}{4\pi} \frac{ds}{R^2}$$

$$B = \int \frac{\mu_0 I}{4\pi} \frac{ds}{R^2}$$

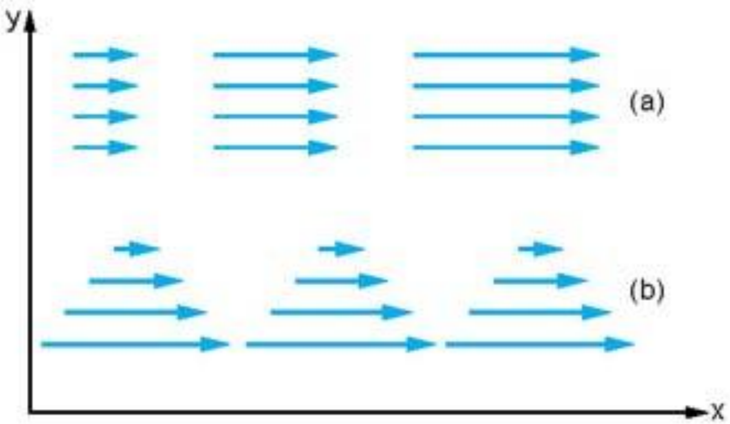
$$B = \frac{\mu_0 I}{4\pi R^2} \int ds$$

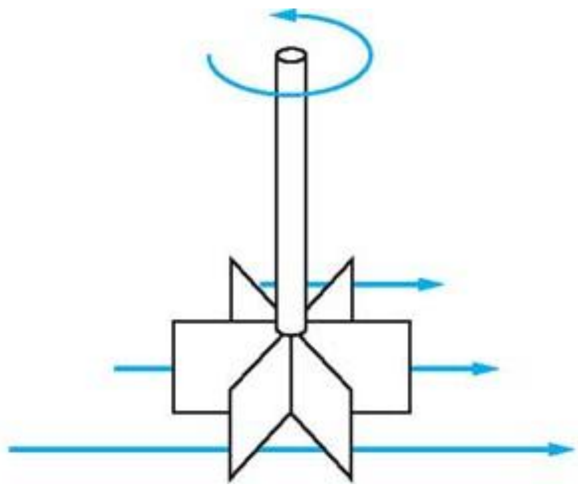
$$B = \frac{\mu_0 I}{4\pi R^2} \int R \, d\theta$$

$$B = \frac{\mu_0 I}{4\pi R} \int d\theta$$

$$B = \frac{\mu_0 I}{4\pi R} \theta$$

Physical view of curl





a) Field lines indicating divergence
b) Field lines indicating curl

A simple way to see the direction of curl using right hand rule

Stokes's Theorem

- *Stokes's Theorem* relates a closed line integral into a surface integral

$$\oint \overrightarrow{H} \cdot d\overrightarrow{L} = \int \left(\nabla \times \overrightarrow{H}\right) \cdot d\overrightarrow{S}$$

Magnetic flux density, B

- *Magnetic flux density* $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ in the free space by
- Magnetic flux ϕ(units of Webers) passing through a surface is found by

$$\vec{B} = \mu_0 \vec{H}$$ Weber/m^2 or Tesla (T)

1 Tesla = 10,000 Gauss.

where μ_0 is the *free space permeability*, given in units of henrys per meter, or

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m.}$$

$$\phi = \int \vec{B} \cdot d\vec{S}$$

Gauss's law for magnetic fields

$$\oint \vec{B} \cdot d\vec{S} = 0$$

or

$$\nabla \cdot \vec{B} = 0.$$

EX1 A solid conductor of circular cross section is made of a homogeneous nonmagnetic material. If the radius *a* = 1 mm, the conductor axis lies on the *z* axis, and the total current in the direction $\hat{a}_z$ is 20 A, find

a) H_ϕ at $\rho = 0.5$ mm

b) B_ϕ at $\rho = 0.8$ mm

c) The total magnetic flux per unit length inside the conductor

Maxwell's equations for static fields

Integral form	Differential form
$\oint \vec{D} \cdot d\vec{S} = Q_{enc}$	$\nabla \cdot \vec{D} = \rho_v$
$\oint \vec{B} \cdot d\vec{S} = 0$	$\nabla \cdot \vec{B} = 0$
$\oint \vec{E} \cdot d\vec{L} = 0$	$\nabla \times \vec{E} = 0$
$\oint \vec{H} \cdot d\vec{L} = I_{enc}$	$\nabla \times \vec{H} = J$

The scalar and vector magnetic potentials (1)

- **Scalar magnetic potential (*V_m*)** $\vec{E} = -\nabla V$ is the simple practical concept to determine the electric field. Similarly, the scalar magnetic potential, V_m, is defined to relate to the magnetic field $\vec{H}$ but there is no physical interpretation.

Assume

$$\vec{H} = -\nabla V_m$$

$$\nabla \times \vec{H} = \vec{J} = \nabla \times (-\nabla V_m) = 0$$

To make the above statement true, ***J* = 0**.

The scalar and vector magnetic potentials (2)

From $$\nabla \cdot \vec{B} = \mu_0 \nabla \cdot \vec{H} = 0$$

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

$$\therefore \nabla^2 V_m = 0$$

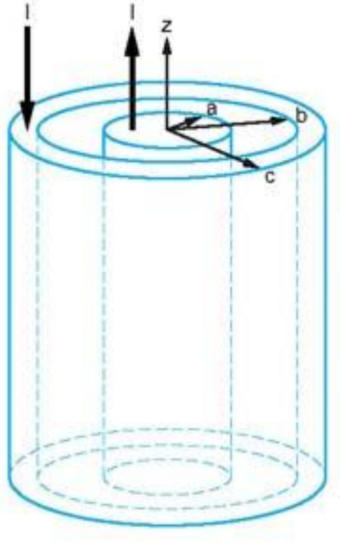
Laplace's equation

This equation's solution to determine the potential field requires that the potential on the boundaries is known.

The scalar and vector magnetic potentials (3)

- The difference between *V* (electric potential) and V_m (scalar magnetic potential) is that the electric potential is a function of the positions while there can be many V_m values for the same position.

$$\oint \vec{H} \cdot d\vec{L} = I_{enc}$$



The scalar and vector magnetic potentials (4)

While for the electrostatic case

$$\nabla \times \vec{E} = 0$$

$$\oint \vec{E} \cdot d\vec{L} = 0$$

$$V_{ab} = -\int_{b}^{a} \vec{E} \cdot d\vec{L}$$ does not depend on path.

The scalar and vector magnetic potentials (5)

- **Vector magnetic potential (*A*)** is useful to find a magnetic filed for antenna and waveguide.

From $$\nabla \cdot \vec{B} = 0$$

Let assume $$\vec{B} = (\nabla \times \vec{A})$$

so $$\nabla \cdot (\nabla \times \vec{A}) = 0$$

and $$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}$$

$$\nabla \times \vec{H} = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A} = \vec{J} \neq 0$$

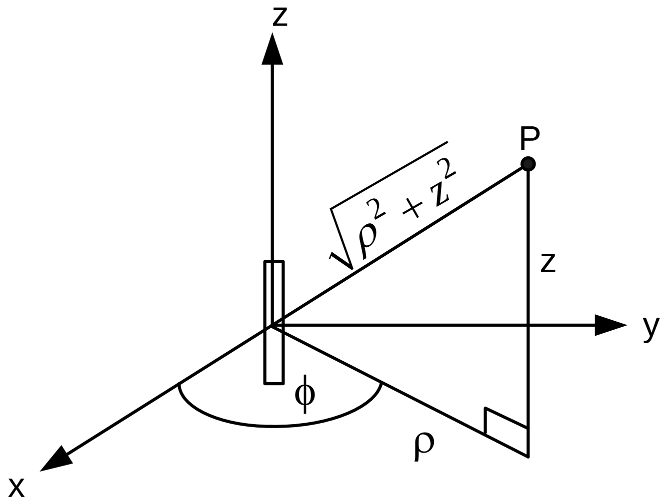
The scalar and vector magnetic potentials (6)

- It is simpler to use the vector magnetic potential to determine the magnetic field. By transforming from Bio-savart law, we can write

$$A = \oint \frac{\mu_0 I d\vec{L}}{4\pi R}.$$

The differential form $$d\vec{A} = \frac{\mu_0 I d\vec{L}}{4\pi R}.$$

Ex: Determine the magnetic field from the infinite length line of current using the vector magnetic potential



$$d\vec{L} = dz\hat{a}_z$$

Find $\vec{A}$ at point P(ρ, ϕ, z)

$$d\vec{A} = \frac{\mu_0 I dz \hat{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$

then $d\vec{H} = \frac{1}{\mu_0}\nabla \times d\vec{A} = \frac{1}{\mu_0}\left(-\frac{\partial dA_z}{\partial \rho}\right)\hat{a}_\phi$

$$d\vec{H} = \frac{I\rho dz}{4\pi\left(\rho^2 + z^2\right)^{3/2}}\hat{a}_\phi$$

Vector magnetic potential for other current distributions

- For current sheet

$$A = \iint_S \frac{\mu_0 \vec{K} dS}{4\pi R}$$

- For current volume

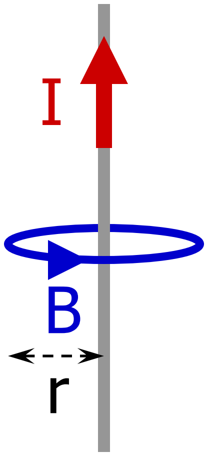
$$A = \iiint_{vol} \frac{\mu_0 \vec{J} dv}{4\pi R}$$

Magnetic Field of a Current-Carrying Wire

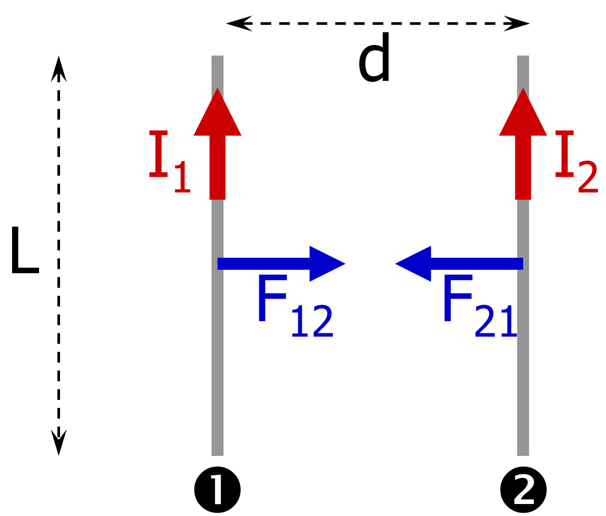
It is experimentally observed that parallel wires exert forces on each other when current flows.

We showed that a long straight wire carrying a current I gives rise to a magnetic field B at a distance r from the wire given by

$$B = \frac{\mu_0 I}{2\pi r}$$



The magnetic field of one wire exerts a force on a nearby current-carrying wire.



The magnitude of the force depends on the two currents, the length of the wires, and the distance between them.

$$F = \frac{\mu_0 I_1 I_2 L}{2\pi d}$$

This is NOT a starting equation

The wires are electrically neutral, so this is not a Coulomb force.

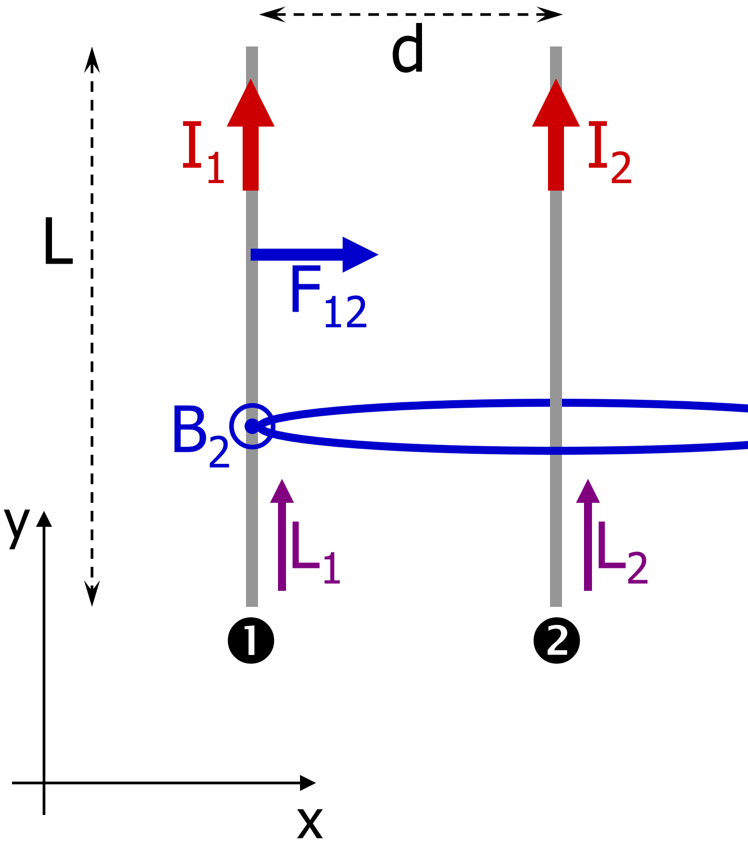
Example: use the expression for B due to a current-carrying wire to calculate the force between two current-carrying wires.

$$\vec{F}_{12} = I_1 \vec{L}_1 \times \vec{B}_2$$

$$\hat{B}_2 = \frac{\mu_0 I_2}{2\pi d} \hat{k}$$

$$\vec{F}_{12} = I_1 L \hat{j} \times \frac{\mu_0 I_2}{2\pi d} \hat{k}$$

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2 L}{2\pi d} \hat{i}$$



The force per unit length of wire is $\frac{\vec{F}_{12}}{L} = \frac{\mu_0 I_1 I_2}{2\pi d} \hat{i}$.

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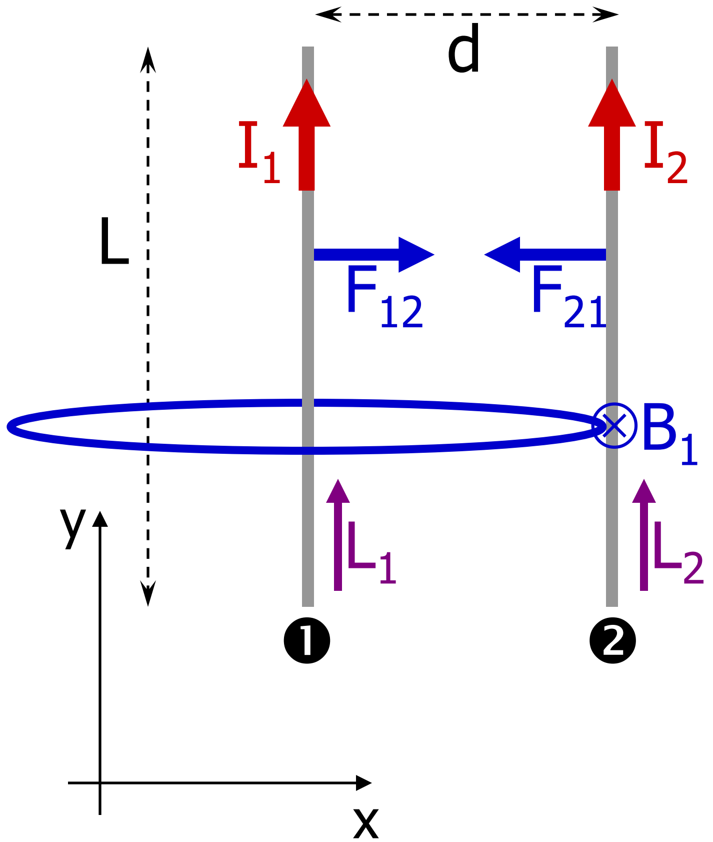
$$\vec{F}_{21} = I_2\vec{L}_2 \times \vec{B}_1$$

$$\vec{B}_1 = -\frac{\mu_0 I_1}{2\pi d}\hat{k}$$

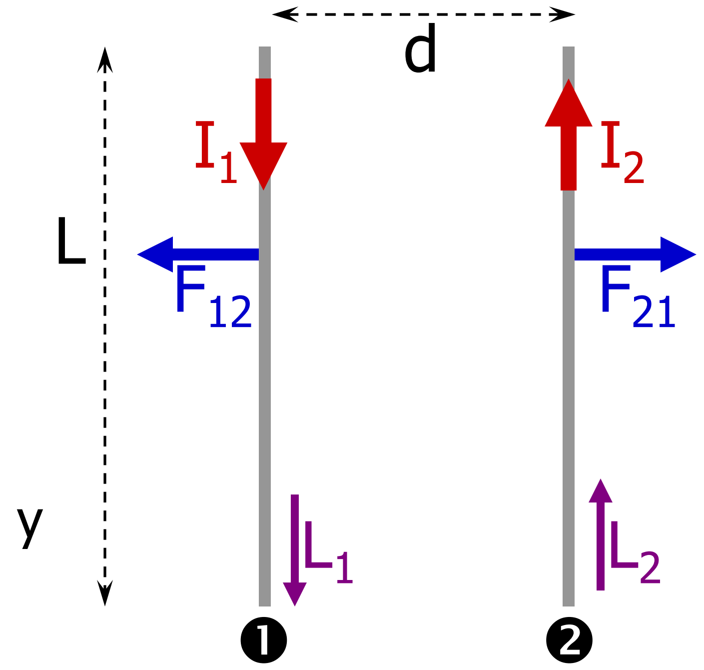
$$\vec{F}_{21} = I_2 L\hat{j} \times \left(-\frac{\mu_0 I_1}{2\pi d}\hat{k}\right)$$

$$\vec{F}_{21} = -\frac{\mu_0 I_1 I_2 L}{2\pi d}\hat{i}$$



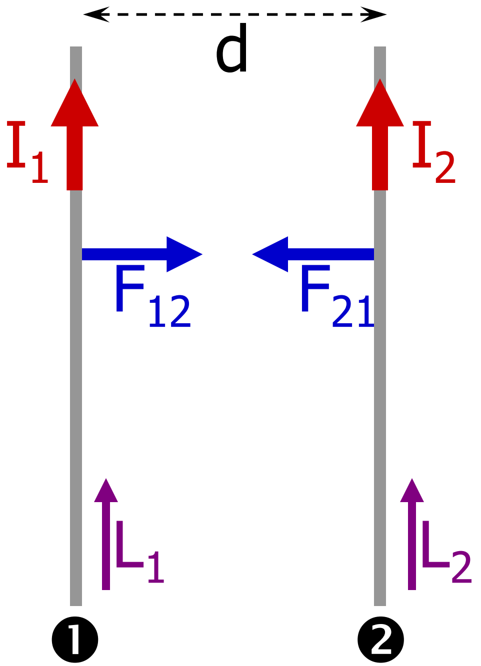
The force per unit length of wire is $\frac{\vec{F}_{21}}{L} = -\frac{\mu_0 I_1 I_2}{2\pi d}\hat{i}$.

If the currents in the wires are in the opposite direction, the force is repulsive.



$$F_{12} = F_{21} = \frac{\mu_0 I_1 I_2 L}{2\pi d}$$

$$F_{12} = F_{21} = \frac{4\pi \times 10^{-7} I_1 I_2 L}{2\pi d} = 2 \times 10^{-7} I_1 I_2 \frac{L}{d}$$

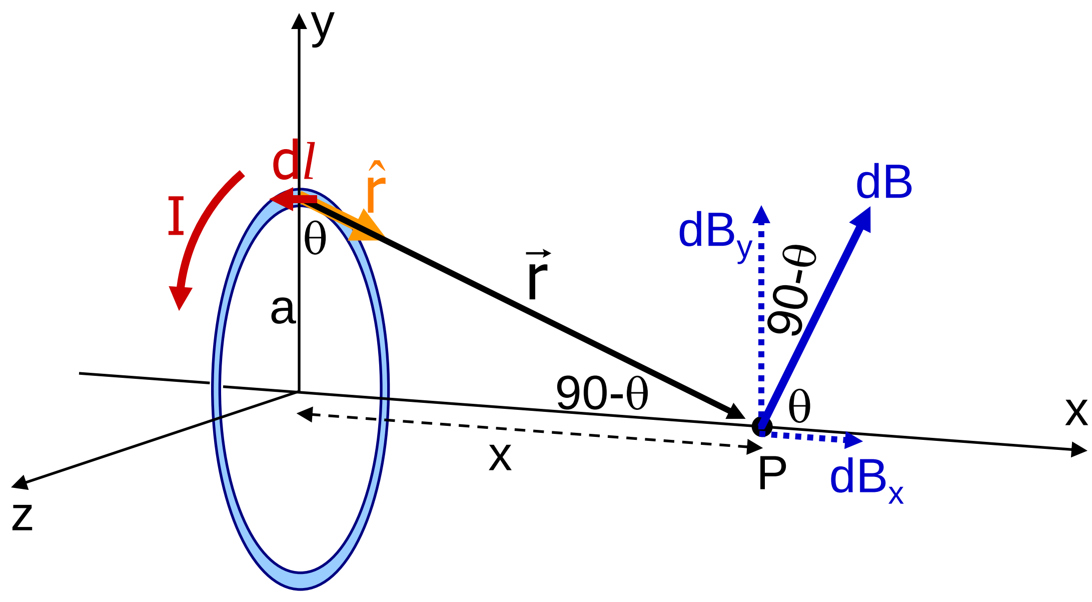


The official definition of the Ampere: 1 A is the current that produces a force of 2×10^{-7} N force per meter of length between two long parallel wires placed 1 meter apart in empty space.

Magnetic Field of a Current Loop

A circular ring of radius a carries a current I as shown. Calculate the magnetic field at a point P along the axis of the ring at a distance x from its center.

Complicated diagram! You are supposed to visualize the ring lying in the yz plane.



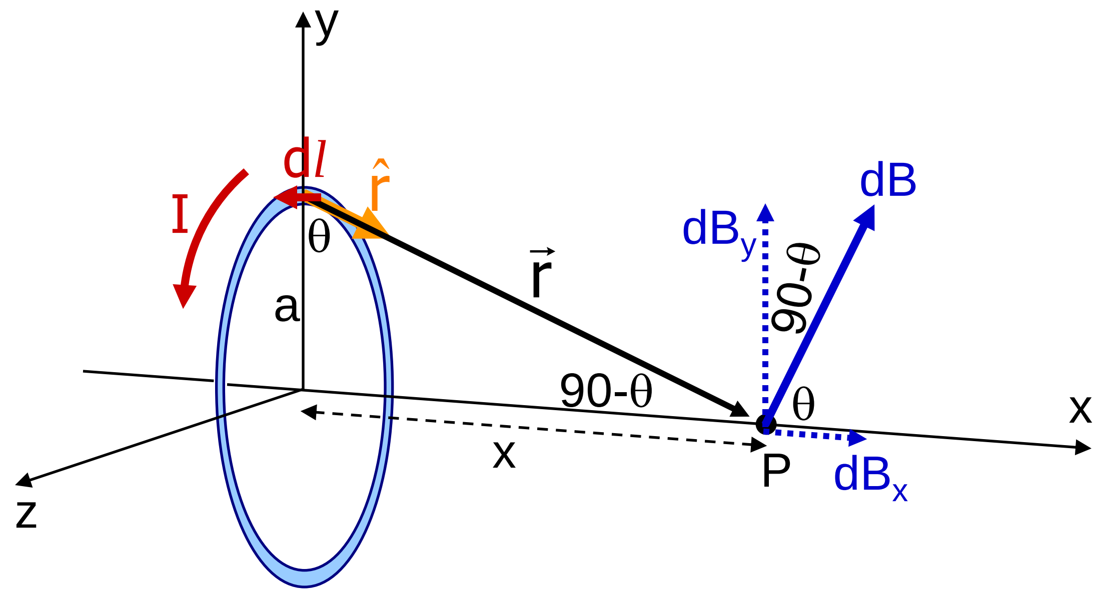
$d\vec{l}$ is in the yz plane. $\hat{r}$ is in the xy plane and is perpendicular to $d\vec{l}$. Thus $\left|d\vec{\ell}\times\hat{r}\right| = d\ell$.

Also, $d\vec{B}$ must lie in the xy plane (perpendicular to $d\vec{l}$) and is perpendicular to $\vec{r}$.

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{\ell} \times \hat{r}}{r^2}$$

$$dB = \frac{\mu_0}{4\pi}\frac{I\, d\ell}{r^2}$$

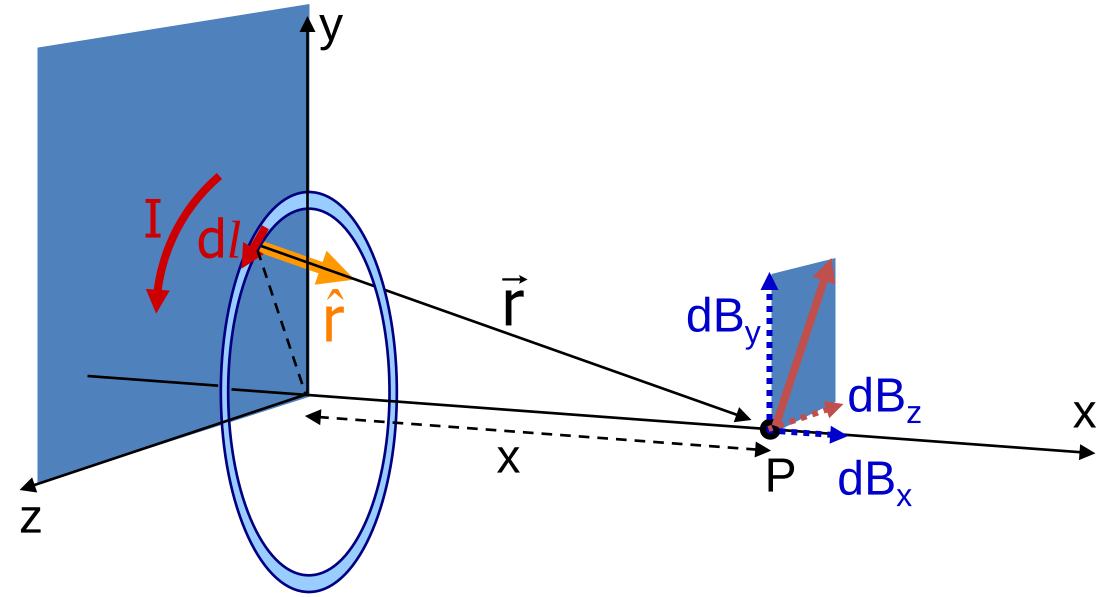
$$dB = \frac{\mu_0}{4\pi}\frac{I\, d\ell}{\left(x^2 + a^2\right)}$$



$$dB_x = \frac{\mu_0}{4\pi}\frac{I\, d\ell}{\left(x^2 + a^2\right)}\cos\theta = \frac{\mu_0}{4\pi}\frac{I\, d\ell}{\left(x^2 + a^2\right)}\frac{a}{\left(x^2 + a^2\right)^{1/2}}$$

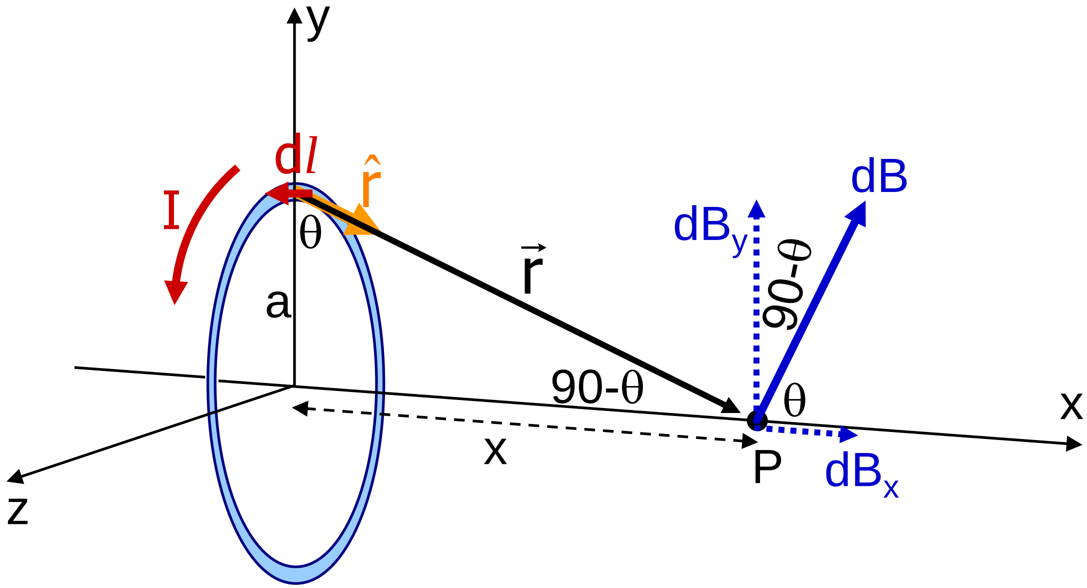
$$dB_y = \frac{\mu_0}{4\pi}\frac{I\, d\ell}{\left(x^2 + a^2\right)}\sin\theta = \frac{\mu_0}{4\pi}\frac{I\, d\ell}{\left(x^2 + a^2\right)}\frac{x}{\left(x^2 + a^2\right)^{1/2}}$$

By symmetry, B_y will be 0. Do you see why?



When $d\vec{l}$ is not centered at z=0, there will be a z-component to the magnetic field, but by symmetry B_z will still be zero.

$$dB_x = \frac{\mu_0}{4\pi} \frac{I\, a\, d\ell}{\left(x^2 + a^2\right)^{3/2}}$$



$$B_x = \int_{ring} dB_x$$

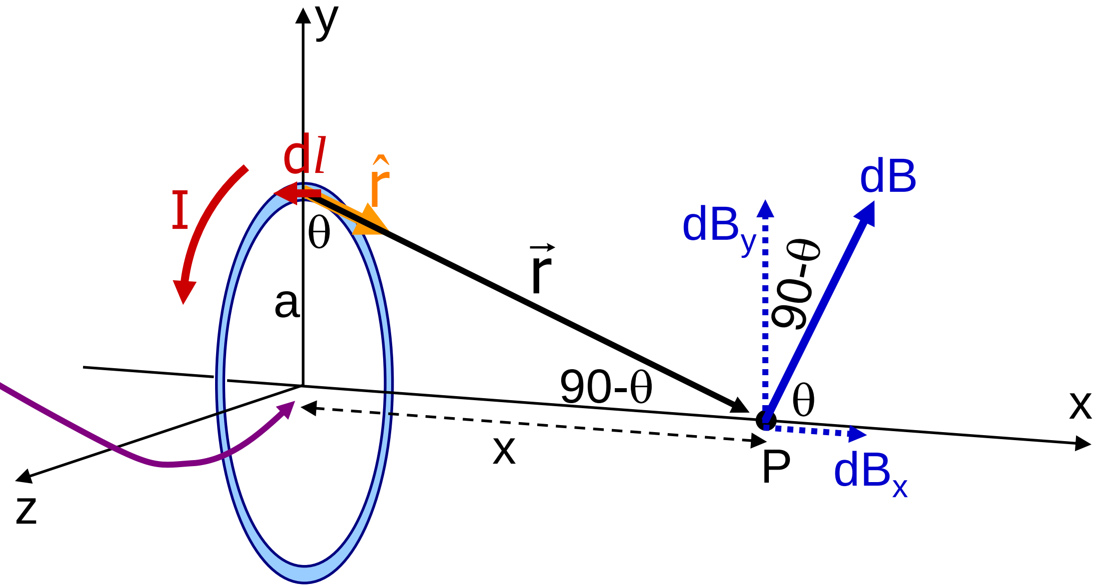
I, x, and a are constant as you integrate around the ring!

$$B_x = \frac{\mu_0}{4\pi} \frac{I\, a}{\left(x^2 + a^2\right)^{3/2}} \int_{ring} d\ell = \frac{\mu_0}{4\pi} \frac{I\, a}{\left(x^2 + a^2\right)^{3/2}} 2\pi a$$

$$B_x = \frac{\mu_0\, I\, a^2}{2\left(x^2 + a^2\right)^{3/2}}$$

Pradeep Singla

At the center of the ring, x=0.



$$B_{x,center} = \frac{\mu_0 \, I \, a^2}{2\left(a^2\right)^{3/2}}$$

$$B_{x,center} = \frac{\mu_0 \, I \, a^2}{2a^3} = \frac{\mu_0 \, I}{2a}$$

For N tightly packed concentric rings (a coil)...

$$B_{x,center} = \frac{\mu_0 \, N \, I}{2a}$$